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## Kicked rotator for a spin- $\frac{1}{2}$ particle

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**Abstract.** The kicked rotator for a spin- $\frac{1}{2}$  particle is introduced and mapped to a tight binding model for a spin- $\frac{1}{2}$  particle. The influence of time reversal invariance is explained and the symplectic kicked rotator is defined. It is shown that for special values of parameters a resonance effect leads to delocalised eigenfunctions of the spin- $\frac{1}{2}$  rotator in the same way as in the case of the well known spin-0 kicked rotator. The infinite-dimensional eigenvalue problem can then be mapped to a finite-dimensional one. Spectral properties are investigated: the maximal degree of level repulsion, depending on the anti-unitary symmetries of the dynamics, and the Shannon entropy of the eigenvectors, measuring their localisation properties, which shows a scaling behaviour similar to the one recently found for the spin-0 kicked rotator.

### 1. Introduction

The kicked quantum rotator was launched a decade ago (Casati *et al* 1979). Since then it kept surprising us with new and unforeseen features: dynamic localisation with quasiperiodic dependence of the energy on time or resonant behaviour with quadratic energy growth in time—depending on the choice of the parameters—both in contradiction to the deterministic diffusion of the classical kicked rotator with its linear energy growth; the formal equivalence to the tight binding model with diagonal disorder, showing Anderson localisation (Fishman *et al* 1982); the possibility of breaking the dynamic localisation by weak dissipation (Dittrich and Graham 1988) with the seemingly paradoxical effect that weak dissipation leads to unbounded growth of energy. Recently, scaling properties of the quantum rotator have been found and investigated (Izrailev 1988, Fishman *et al* 1988, Casati *et al* 1989b). Especially in the above mentioned case of resonant behaviour with extended eigenstates and continuous spectrum when Bloch's theorem applies, the eigenvectors, upon changing some perturbation parameter  $K$ , show an interesting additional delocalisation transition within one lattice period. Deep similarities between this scaling and the finite size scaling of the Anderson transition were recognised by Casati *et al* (1989b).

All this seems not to exhaust the interesting features of the model. As the localisation–delocalisation transition is accompanied by a drastic change of the statistical features of the spectrum of the rotator, scaling behaviour of the spectrum itself was supposed and is under investigation. Upon increasing  $K$  the nearest-neighbour spacing distribution of the eigenvalues of the rotator for fixed Bloch number  $a$ , for example, shows a characteristic transition from (nearly) Poissonian type, with only

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negligible repulsion in the localised case, to Wigner type with strong repulsion in the completely delocalised case. The latter can be described by random matrix theory (RMT) taking ensembles of unitary matrices of certain rank and (anti-unitary) symmetry. Three ensembles are generic, the so-called circular orthogonal, unitary and symplectic ensembles (COE, CUE and CSE, respectively) which have been introduced by Dyson (1962). The most prominent difference between these three ensembles is their different degree of eigenphase repulsion, namely linear (COE), quadratic (CUE) and quartic (CSE). Realisations for all three classes have already been found in the form of strongly kicked spin dynamics with non-integrable classical limit (Scharf *et al* 1988) (see also Courier and Grammaticos (1989) for autonomous realisations). But these spin systems do not show any kind of Anderson localisation that might change the behaviour in a scale invariant way. Therefore the construction of modified kicked rotators which show quadratic and quartic level repulsion in the limit of complete delocalisation became an interesting goal. This has partially been fulfilled by a modified kicked rotator without (generalised) time-reversal invariance (Izrailev 1986), which shows quadratic level repulsion. A transition to linear repulsion upon decreasing the strength of the kick was observed stemming from a partial localisation of the eigenfunctions, and finally the Poissonian case was approached for strongly localised eigenfunctions.

To observe eigenvalue repulsion of quartic degree and to see the full localisation induced transition from quartic over quadratic and linear repulsion to Poissonian behaviour it is known that the system has to be invariant under time reversal generated by a (anti-unitary) time-reversal operator  $\mathcal{T}$  that squares to  $-1$ . This is only possible for a system with half-integer spin. An immediate consequence of  $\mathcal{T}$ -invariance with  $\mathcal{T}^2 = -1$  is Kramers' degeneracy, namely: each eigenvalue of the dynamics is twofold degenerate. Another motivation for studying the kicked rotator of a spin- $\frac{1}{2}$  particle comes from the equivalence of the standard (spin-0) kicked rotator to a tight binding model for a spin-0 particle. As it is known that tight binding models for spin- $\frac{1}{2}$  particles may show new and interesting phenomena (Zanon and Pichard 1988) such as, for example, the Anderson transition in two dimensions (Evangelou and Ziman 1988), the question arises whether a kicked rotator for a spin- $\frac{1}{2}$  particle can be constructed and mapped to the corresponding tight binding model. Therefore the time has come for the spin- $\frac{1}{2}$  kicked rotator.

In §2 I define the spin- $\frac{1}{2}$  kicked rotator Hamiltonian and turn to the resonant case, which allows one to reduce the original problem with infinite-dimensional Hilbert space to a finite-dimensional one by fixing a Bloch number. In §3 the general, not necessarily resonant, problem is mapped to a tight binding model via the trick of Fishman *et al* (1982) with slight modifications. In §4 I investigate the consequences of generalised time-reversal invariances and proceed to construct the symplectic kicked rotator. In §5 the resonant symplectic rotator is investigated whose eigenvalue problem can be solved by diagonalising a finite-dimensional unitary matrix. Then it is shown how additional unitary symmetries of the propagator of the dynamics, which square to  $-1$  and commute with the anti-unitary time-reversal operator  $\mathcal{T}$ , influence the maximal degree of level repulsion in the case of completely delocalised eigenfunctions. The role played by the Bloch number is discussed. In §6 I present numerical results for the case of strongly delocalised eigenfunctions. In §7 the transition to exponentially localised eigenfunctions is discussed and evidence for scaling behaviour in the spin- $\frac{1}{2}$  kicked rotator is presented. This paper closes with a discussion and an outlook on related questions.

## 2. Kicked rotator for a spin- $\frac{1}{2}$ particle

The Hamiltonian for the standard kicked rotator (Casati *et al* 1979) has the form

$$H(t) = \frac{p^2}{2} + KV(\theta) \sum_{n=-\infty}^{+\infty} \delta(t - nT) \quad (2.1)$$

with  $\theta$  being a  $2\pi$ -periodic angle,  $p$  the conjugate (angular) momentum,  $V(\theta) = V(\theta + 2\pi)$  the kicking potential. The unitary propagator  $U$  that generates the stroboscopic quantum map is gained upon integrating  $H(t)$  over one period  $T$ . Starting the integration halfway between two successive kicks brings  $U$  in a symmetric form which is well suited for symmetry consideration:

$$U = \exp\left(-i\frac{Tp^2}{4\hbar}\right) \exp\left(-i\frac{KV}{\hbar}\right) \exp\left(-i\frac{Tp^2}{4\hbar}\right). \quad (2.2)$$

The operator  $U$  generates the quantum map

$$\psi_{t+1} = U\psi_t. \quad (2.3)$$

Since, in the ' $\theta$  representation' (but see Peierls 1979),  $p$  is of the form  $-i\hbar\partial_\theta$  it can be seen that  $U$  contains only two parameters explicitly (besides other ones hidden in  $V$ ), namely  $\tau = \hbar T$  and  $k = K/\hbar$ . Upon choosing  $T$  as the unit of time we have  $\tau = \hbar$  and  $k\tau = K$ . The classical limit is achieved by  $\tau \rightarrow 0$ ,  $k \rightarrow \infty$  upon fixing  $k\tau = K$ . For  $K \simeq 5$  the corresponding classical map shows well developed chaos and deterministic diffusion (Lichtenberg and Lieberman 1983). The behaviour of the quantum map is also well investigated (Chirikov *et al* 1981, Eckhardt 1988).

To proceed to the spin- $\frac{1}{2}$  case we have to choose a spinor representation for the Hamiltonian. An obvious choice is

$$H(t) = \frac{p^2}{2} \mathbf{1} + K(V_0(\theta)\mathbf{1} + V(\theta)\boldsymbol{\sigma}) \sum_{n=-\infty}^{+\infty} \delta(t - nT) \quad (2.4)$$

where  $\mathbf{1}$  denotes the  $(2 \times 2)$ -unit matrix,  $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$  the vector of the three Pauli spin matrices and  $V(\theta) = (V_1, V_2, V_3)$  a vector composed of three potentials being  $2\pi$ -periodic in  $\theta$ , which is the case for  $V_0(\theta)$ , too. Finally  $V(\theta)\boldsymbol{\sigma}$  denotes the 'scalar product'  $V_1\sigma_1 + V_2\sigma_2 + V_3\sigma_3$ . The unitary propagator in the symmetric form can be written down immediately, setting  $\tau = \hbar T$ ,  $k = K/\hbar$  and  $T = 1$  as in the spin-0 case:

$$U = \exp\left(-i\frac{p^2}{4\tau}\right) \exp[-ik(V_0 + V\boldsymbol{\sigma})] \exp\left(-i\frac{p^2}{4\tau}\right) \quad (2.5)$$

suppressing the unit matrices. Using well known properties of the Pauli spin matrices, this can be written as

$$U = \exp\left(-i\frac{p^2}{4\tau}\right) \exp(-ikV_0(\theta)) \left\{ \mathbf{1} \cos(k|V(\theta)|) - i\frac{V(\theta)\boldsymbol{\sigma}}{|V(\theta)|} \sin(k|V(\theta)|) \right\} \\ \times \exp\left(-i\frac{p^2}{4\tau}\right) \quad (2.6)$$

with  $|V(\theta)| = \sqrt{V_1(\theta)^2 + V_2(\theta)^2 + V_3(\theta)^2}$ .

A numerical approximation to the iteration (2.3) with the help of a fast Fourier transform can then be done as in the case of the spin-0 kicked rotator, barring some minor modifications. But as we are interested in solving the eigenvalue equation we now turn to the resonant case which allows the setting up of a unitary matrix of finite rank which can then be diagonalised by standard methods.

First we introduce the  $p$  eigenbasis, which is also the eigenbasis of  $U(k=0)$ :

$$p|\psi\rangle = p_\psi|\psi\rangle \quad (2.7a)$$

and in the ' $\theta$  representation'  $\langle\theta|\psi\rangle = \psi(\theta) = \psi(\theta + 2\pi)$ :

$$\langle\theta|p|\psi\rangle = -i\hbar\hat{c}_\theta\psi(\theta) = p_\psi\psi(\theta). \quad (2.7b)$$

This leads to the eigenstates  $|\psi_n\rangle = |n\rangle$  with

$$\langle\theta|n\rangle = \psi_n(\theta) = \exp(in\theta) \quad n \text{ integer} \quad (2.7c)$$

as is well known. If we choose a value for  $\tau$  being commensurate with  $\pi$  in the form

$$\tau = 4\pi M/N \quad N, M \text{ relatively prime} \quad (2.8)$$

we find in the  $p$  representation

$$\langle m+N|U^{\rho\sigma}|n+N\rangle = \exp[-\frac{1}{4}i\tau(m+N)^2]\langle m+N|F^{\rho\sigma}|n+N\rangle \exp[-\frac{1}{4}i\tau(n+N)^2] \quad (2.9)$$

with spinor indices  $\rho, \sigma = \pm 1$  and  $\langle m+N|F^{\rho\sigma}|n+N\rangle \equiv \hat{F}_{n-m}^{\rho\sigma}$  being the Fourier transform of

$$F(\theta)^{\rho\sigma} = \exp(-ikV_0(\theta)) \left\{ \mathbf{1} \cos(k|V(\theta)|) - i \frac{V(\theta)\sigma}{|V(\theta)|} \sin(k|V(\theta)|) \right\}^{\rho\sigma} \quad (2.10)$$

the  $\rho\sigma$  component of a  $(2 \times 2)$  matrix function of  $\theta$ . Using property (2.8) of  $\tau$  we find

$$\langle m+N|U^{\rho\sigma}|n+N\rangle = \langle m|U^{\rho\sigma}|n\rangle \equiv U_{mn}^{\rho\sigma}. \quad (2.11)$$

The apparent periodicity of  $U$  in the  $p$  representation can now be used to cast the eigenvalue problem in finite-dimensional form. The similarity with the resonant case for the spin-0 kicked rotator is obvious, and the construction of a corresponding unitary  $(2N \times 2N)$  matrix can be achieved by following the lines given by Chang and Shi (1987) for the spin-0 case. Therefore I only sketch the procedure. First we rewrite the eigenvalue problem in spinor form

$$U^{\rho\sigma}|\psi^\sigma\rangle = e^{-i\phi}|\psi^\rho\rangle \quad (2.12)$$

where  $\rho, \sigma = \pm 1$  and summation over repeated spinor indices is assumed. As  $U$  is periodic in the  $p$  representation, Bloch's theorem holds and the eigenstates  $\{|\psi^\sigma\rangle\}_{\sigma=\pm 1}$  have the property

$$\langle m+N|\psi^\sigma\rangle \equiv \psi_{m+N}^\sigma = e^{-ia}\psi_m^\sigma \quad (2.13)$$

with the Bloch number  $a$  ( $-\pi \leq a < \pi$ ) giving the condition at the boundary of the fundamental lattice period of length  $N$  in the  $p$  representation. We now define

$$U(a)_{rs}^{\rho\sigma} = \sum_{l=-\infty}^{+\infty} U_{r,s+Nl}^{\rho\sigma} e^{-ial} \tag{2.14}$$

and notice that

$$\sum_{s=1}^N U(a)_{rs}^{\rho\sigma} \psi_s^\sigma = \sum_{s=1}^N \sum_{l=-\infty}^{+\infty} U_{r,s+Nl}^{\rho\sigma} \psi_{s+Nl}^\sigma = \sum_{n=-\infty}^{+\infty} U_{rn}^{\rho\sigma} \psi_n^\sigma = e^{-i\phi} \psi_r^\rho. \tag{2.15}$$

This shows the equivalence of the original infinite-dimensional eigenvalue problem to a set of finite-dimensional ones indexed by the continuous parameter  $a$ . Diagonalising the unitary ( $2N \times 2N$ ) matrix  $U(a)$  for fixed Bloch number  $a$  solves the original eigenvalue problem. To calculate  $U(a)_{rs}^{\rho\sigma}$  explicitly we use the following representation of the periodic  $\delta$ -function

$$\sum_{l=-\infty}^{+\infty} \exp[iil(N\theta - a)] = \frac{2\pi}{N} \sum_{j=-\infty}^{+\infty} \delta\left(\theta - \frac{2\pi j + a}{N}\right) \tag{2.16}$$

which can be easily checked by the means of Fourier transformation. We then find for the components of  $U(a)$

$$\begin{aligned} U(a)_{rs}^{\rho\sigma} &= \sum_{l=-\infty}^{+\infty} \exp[-\frac{1}{4}i\tau(r^2 + s^2)] \langle r | F^{\rho\sigma} | s + Nl \rangle e^{-ial} \\ &= \exp[-\frac{1}{4}i\tau(r^2 + s^2)] \sum_{l=-\infty}^{+\infty} \frac{1}{2\pi} \int_0^{2\pi} d\theta \exp(-ir\theta) F(\theta)^{\rho\sigma} \\ &\quad \times \exp [i(s + Nl)\theta - ial]. \end{aligned} \tag{2.17}$$

Finally using (2.16) and performing the integral leads to

$$U(a)_{rs}^{\rho\sigma} = \exp[-\frac{1}{4}i\tau(r^2 + s^2)] \frac{1}{N} \sum_{j=1}^N \exp [i(s - r)\theta_j] F(\theta_j)^{\rho\sigma} \tag{2.18}$$

with  $\theta_j = (2\pi j + a)/N$  and  $F(\theta)^{\rho\sigma}$  defined by (2.10). The angle  $b = a/N$  is restricted to  $-\pi/N \leq b < \pi/N$ . The diagonalisation of the matrix  $U(a)$  is now a straightforward numerical task.

### 3. Mapping to the tight binding model for a spin- $\frac{1}{2}$ particle

Fishman *et al* (1982) mapped the spin-0 kicked rotator to a tight binding model for a spin-0 particle. This can be done for the spin- $\frac{1}{2}$  rotator in an analogous way. Again I only sketch the derivation, stressing the necessary modifications in comparison with the spin-0 case. Simplifying notation I write

$$V = V_0 \mathbf{1} + V \boldsymbol{\sigma}. \tag{3.1}$$

Now it is more convenient to look at an asymmetric propagator  $U$  being unitarily equivalent to the one defined by (2.5)

$$U = e^{-ikV} U_0 \quad U_0 = \exp\left(-i\frac{p^2}{2\tau}\right) \mathbf{1} = \exp(-iH_0)\mathbf{1}. \quad (3.2)$$

Introducing state vectors  $|u_{\pm}^{\sigma}\rangle$  and eigenphases  $\phi$  by

$$U^{\rho\sigma}|u_{+}^{\sigma}\rangle = e^{-i\phi}|u_{+}^{\rho}\rangle \quad (3.3a)$$

$$|u_{-}^{\rho}\rangle = [e^{ikV}]^{\rho\sigma}|u_{+}^{\sigma}\rangle = \exp[i(\phi - H_0)]|u_{+}^{\sigma}\rangle \quad (3.3b)$$

and the symmetrised state

$$|u^{\sigma}\rangle \equiv \frac{1}{2}(|u_{-}^{\sigma}\rangle + |u_{+}^{\sigma}\rangle) \quad (3.3c)$$

and writing the unitary kicking operator in the form

$$e^{\pm ikV} = \frac{\mathbf{1} \mp iW}{\mathbf{1} \pm iW} \quad (3.4)$$

(the operator  $W$  will be constructed explicitly later on), we can express  $|u^{\rho}\rangle$  with the help of  $|u_{-}^{\rho}\rangle$ , namely

$$|u^{\rho}\rangle = \frac{1}{2}(\mathbf{1} + e^{-ikV})^{\rho\sigma}|u_{-}^{\sigma}\rangle = \left(\frac{1}{\mathbf{1} - iW}\right)^{\rho\sigma}|u_{-}^{\sigma}\rangle \quad (3.5a)$$

and with  $|u_{+}^{\sigma}\rangle$ :

$$|u^{\rho}\rangle = \frac{1}{2}(\mathbf{1} + e^{ikV})^{\rho\sigma}|u_{+}^{\sigma}\rangle = \left(\frac{1}{\mathbf{1} + iW}\right)^{\rho\sigma}|u_{+}^{\sigma}\rangle. \quad (3.5b)$$

Using the preceding equations yields

$$(\mathbf{1} - iW)^{\rho\sigma}|u^{\sigma}\rangle = |u_{-}^{\rho}\rangle = \exp[i(\phi - H_0)]|u_{+}^{\rho}\rangle = \exp[i(\phi - H_0)](\mathbf{1} + iW)^{\rho\sigma}|u^{\sigma}\rangle. \quad (3.6)$$

And finally

$$i\frac{1 - \exp[i(\phi - H_0)]}{1 + \exp[i(\phi - H_0)]}|u^{\rho}\rangle + W^{\rho\sigma}|u^{\sigma}\rangle = 0. \quad (3.7)$$

The connection with the tight binding model becomes transparent upon writing the last equation in the  $p$  representation

$$\tan\left(\frac{\phi}{2} - \frac{\hbar m^2}{4}\right)\langle m|u^{\rho}\rangle + \sum_n \langle m|W^{\rho\sigma}|m+n\rangle\langle m+n|u^{\sigma}\rangle = 0 \quad (3.8)$$

and introducing the Fourier transform  $\hat{W}$  of  $W$

$$\hat{W}_n^{\rho\sigma} = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{in\theta} W^{\rho\sigma}(\theta) = \langle m|W^{\rho\sigma}|m+n\rangle. \quad (3.9)$$

Then (3.8) takes the form

$$\tan\left(\frac{\phi}{2} - \frac{\hbar m^2}{4}\right) u_m^\rho + \sum_{n \neq 0} \hat{W}_n^{\rho\sigma} u_{m+n}^\sigma = -\hat{W}_0^{\rho\sigma} u_m^\sigma. \quad (3.10)$$

This is a tight binding equation for a spin- $\frac{1}{2}$  particle.  $\hat{W}_0^{\rho\sigma}$  plays the role of an energy if it is of the form  $\hat{W}_0 = E \mathbf{1}$  with real  $E$ , as is the case for the symplectic model that I investigate in the next section. For  $\hbar = \tau$  being incommensurate with  $\pi$  the term  $\tan(\frac{1}{2}\phi - \frac{1}{4}\hbar m^2)$  represents the diagonal disorder term of the tight binding Hamiltonian. For the resonant case (2.8), on the other hand, these diagonal terms become periodic in  $m$  with period  $N$ . This leads to the existence of Bloch states and a continuous spectrum of  $U$ , as has already been shown in the previous section.

We conclude this section by calculating  $W^{\rho\sigma}$  for a given kicking potential  $V^{\rho\sigma}$ . We invert equation (3.4)

$$W^{\rho\sigma} = -[\tan \frac{1}{2}kV]^{\rho\sigma} = -[\tan(\frac{1}{2}k(V_0 \mathbf{1} + V \boldsymbol{\sigma}))]^{\rho\sigma} \quad (3.11)$$

and note for real  $x_0$  and  $\mathbf{x} = (x_1, x_2, x_3)$  the spectral decomposition of  $x_0 \mathbf{1} + \mathbf{x} \boldsymbol{\sigma}$  is of the form

$$x_0 \mathbf{1} + \mathbf{x} \boldsymbol{\sigma} = \frac{1}{2}(x_0 + |\mathbf{x}|)(\mathbf{1} + \mathbf{x} \boldsymbol{\sigma} |\mathbf{x}|^{-1}) + \frac{1}{2}(x_0 - |\mathbf{x}|)(\mathbf{1} - \mathbf{x} \boldsymbol{\sigma} |\mathbf{x}|^{-1}) \quad (3.12)$$

which leads to

$$\begin{aligned} \tan(x_0 \mathbf{1} + \mathbf{x} \boldsymbol{\sigma}) &= \frac{1}{2} [\tan(x_0 + |\mathbf{x}|) + \tan(x_0 - |\mathbf{x}|)] \mathbf{1} \\ &+ \frac{1}{2} [\tan(x_0 + |\mathbf{x}|) - \tan(x_0 - |\mathbf{x}|)] \mathbf{x} \boldsymbol{\sigma} / |\mathbf{x}| \end{aligned} \quad (3.13)$$

as long as  $|\mathbf{x}| \pm x_0$  is not an odd multiple of  $\pi/2$ . Finally we have

$$\begin{aligned} W &= -\frac{1}{2} \{ \tan[\frac{1}{2}k(V_0 + |V|)] + \tan[\frac{1}{2}k(V_0 - |V|)] \} \mathbf{1} \\ &- \frac{1}{2} \{ \tan[\frac{1}{2}k(V_0 + |V|)] - \tan[\frac{1}{2}k(V_0 - |V|)] \} V \boldsymbol{\sigma} / |V| \end{aligned} \quad (3.14)$$

which completes the mapping to the spin- $\frac{1}{2}$  particle tight binding model. The choice  $V \equiv 0$  decouples the spin-up and spin-down states and leads to degeneracy.  $W$  is then virtually the same as for the spin-0 kicked rotator (Fishman *et al* 1982) with its spin-independent potential.

#### 4. Time reversal symmetry and the symplectic rotator

Now we impose restrictions on the unitary propagator  $U$  for the spin- $\frac{1}{2}$  rotator in the general case (2.5) as well as in the resonant case (2.18), namely in the form of anti-unitary symmetries called generalised time-reversal symmetries (Messiah 1961, Porter 1965). According to the properties of the (common) classical time reversal a quantum time-reversal operator  $\mathcal{T}$  should act on  $\theta$ ,  $p$  and  $\boldsymbol{\sigma}$  in the following way:

$$\mathcal{T} \theta \mathcal{T}^{-1} = \theta \quad \mathcal{T} p \mathcal{T}^{-1} = -p \quad \mathcal{T} \boldsymbol{\sigma} \mathcal{T}^{-1} = -\boldsymbol{\sigma}. \quad (4.1)$$



It follows that  $\mathcal{T}$  is anti-unitary:  $\mathcal{T} \hbar e^{i\theta} \mathcal{T}^{-1} = \mathcal{T} [p, e^{i\theta}] \mathcal{T}^{-1} = -[p, \mathcal{T} e^{i\theta} \mathcal{T}^{-1}]$  which implies that:  $\mathcal{T} e^{i\theta} \mathcal{T}^{-1} = e^{-i\theta}$ . In general it holds that

$$\mathcal{T} c \mathcal{T}^{-1} = c^* \tag{4.2}$$

for arbitrary complex number  $c$ .

After fixing the representation,  $\mathcal{T}$  can be decomposed into a product of a unitary operator and a conjugation operator. If we choose, for example, a representation in which the momentum eigenfunctions are of the form  $e^{in\theta}$ , then the momentum operator is imaginary:  $p = -i\hbar \partial_\theta$  (2.7b) and changes sign under conjugation. For the spin- $\frac{1}{2}$  case we fix a spinor representation by choosing the Pauli spin matrices  $\sigma_1, \sigma_3$  real and  $\sigma_2$  imaginary. Then only  $\sigma_2$  will change sign under conjugation. We summarise the action of the conjugation operator  $\mathcal{K}_\theta$  in this representation:

$$\begin{aligned} \mathcal{K}_\theta \theta \mathcal{K}_\theta &= \theta & \mathcal{K}_\theta p \mathcal{K}_\theta &= -p & \mathcal{K}_\theta \sigma_\nu \mathcal{K}_\theta &= \sigma_\nu & \nu &= 1, 3 \\ \mathcal{K}_\theta \sigma_2 \mathcal{K}_\theta &= -\sigma_2 & \mathcal{K}_\theta c \mathcal{O} \mathcal{K}_\theta &= c^* \mathcal{K}_\theta \mathcal{O} \mathcal{K}_\theta \end{aligned} \tag{4.3}$$

for arbitrary complex number  $c$  and arbitrary operator  $\mathcal{O}$ . Now we can define the time-reversal operator  $\mathcal{T}_\theta$

$$\mathcal{T}_\theta = \exp(\frac{1}{2}i\pi\sigma_2) \mathcal{K}_\theta = \mathcal{K}_\theta \exp(\frac{1}{2}i\pi\sigma_2). \tag{4.4}$$

$\mathcal{T}_\theta$  is anti-linear, as was  $\mathcal{K}_\theta$ , anti-unitary with the inverse  $\mathcal{T}_\theta^{-1} = \exp(-\frac{1}{2}i\pi\sigma_2) \mathcal{K}_\theta$ , and it generates the following time reversal

$$\mathcal{T}_\theta \theta \mathcal{T}_\theta^{-1} = \theta \quad \mathcal{T}_\theta p \mathcal{T}_\theta^{-1} = -p \quad \mathcal{T}_\theta \sigma_\nu \mathcal{T}_\theta^{-1} = -\sigma_\nu \tag{4.5}$$

for all three Pauli spin matrices ( $\nu = 1, 2, 3$ ).

For an autonomous dynamics with Hamiltonian  $H$ , time-reversal invariance with respect to a time-reversal operator  $\mathcal{T}$  ( $\mathcal{T}$ -invariance for short) means

$$\mathcal{T} H \mathcal{T}^{-1} = H. \tag{4.6}$$

As an immediate consequence of this we have: if  $\mathcal{T}$  acts as a conjugation in some representation then the eigenfunctions of  $H$  are real in that representation.

For the unitary propagator generated by  $H$  we then have

$$\mathcal{T} \exp\left(-i\frac{tH}{\hbar}\right) \mathcal{T}^{-1} = \exp\left(i\frac{tH}{\hbar}\right) \tag{4.7}$$

if we treat  $t$  as a real parameter. The time-reversal invariance of a kicked dynamics then poses the following condition on its unitary propagator

$$\mathcal{T} U \mathcal{T}^{-1} = U^\dagger. \tag{4.8}$$

One might easily check that the standard kicked rotator is  $\mathcal{T}_\theta$ -invariant as its propagator (2.2) fulfils condition (4.8) with  $\mathcal{T}_\theta$  defined by (4.4). This time-reversal invariance can be broken immediately by including terms in  $U$  that are linear in  $p$ . But then another  $\mathcal{T}$ -invariance might still exist.

For example we look at the kicked rotator for a charged spin-0 particle in a magnetic field. The coupling with the field produces terms in the free propagator that are linear in  $p$ . We therefore modify the propagator (2.2)

$$\tilde{U} = \exp\left(-i\frac{(p^2 + \gamma p)}{4\hbar}\right) \exp(-ikV(\theta)) \exp\left(-i\frac{(p^2 + \gamma p)}{4\hbar}\right) \quad (4.9)$$

with some real, non-zero parameter  $\gamma$ . Obviously  $\tilde{U}$  is no longer  $\mathcal{T}_\theta$ -invariant. But if we choose a time-reversal operator  $\mathcal{T} = \mathcal{T}_p$  which differs from  $\mathcal{T}_\theta$  only by being a conjugation in the  $p$  representation,  $\tilde{U}$  might nevertheless be  $\mathcal{T}$ -invariant.  $\mathcal{T}_p$  has the properties

$$\mathcal{T}_p \theta \mathcal{T}_p^{-1} = -\theta \quad \mathcal{T}_p p \mathcal{T}_p^{-1} = p \quad \mathcal{T}_p \sigma_v \mathcal{T}_p^{-1} = -\sigma_v \quad (4.10)$$

for  $v = 1, 2, 3$ . As one may immediately check,  $\tilde{U}$  fulfils  $\mathcal{T}_p \tilde{U} \mathcal{T}_p^{-1} = \tilde{U}^\dagger$ , if  $V(-\theta) = V(\theta)$ . Or, in other words,  $\tilde{U}$  is  $\mathcal{T}_p$ -invariant if  $V(\theta)$  is an even function of  $\theta$ .

Over the past few years numerical evidence has grown for the assertion that, if the spectral properties of a unitary propagator  $U$  can be described by RMT, the anti-unitary symmetries, or  $\mathcal{T}$ -invariances, fix which of the three ensembles of unitary random matrices, mentioned in the introduction (namely COE, CUE or CSE) is relevant (Izrailev 1986, 1987, Kuš *et al* 1988, Scharf *et al* 1988). In this respect the time-reversal operator  $\mathcal{T} = \mathcal{T}_p$  is as good as  $\mathcal{T}_\theta$ . Both of them are physically reasonable:  $\mathcal{T}_\theta$  is the ‘standard time-reversal operator’ for a spin- $\frac{1}{2}$  system and for  $\mathcal{T}_p$  only the roles of  $p$  and  $\theta$  are exchanged.

Both time-reversal operators,  $\mathcal{T}_p$  as well as  $\mathcal{T}_\theta$ , have an important property, namely they square to  $-1$ :

$$\begin{aligned} \mathcal{T}^2 &= \exp(\tfrac{1}{2}i\pi\sigma_2)\mathcal{K} \exp(\tfrac{1}{2}i\pi\sigma_2)\mathcal{K} \\ &= \exp(\tfrac{1}{2}i\pi\sigma_2) \exp[-\tfrac{1}{2}i\pi(-\sigma_2)]\mathcal{K}^2 \\ &= \exp(i\pi\sigma_2) = -1. \end{aligned} \quad (4.11)$$

As an immediate consequence we have that for an arbitrary state vector  $\psi$  in Hilbert space the state vector  $\mathcal{T}\psi$  is orthogonal (Messiah 1961, Porter 1965). If  $\psi$  is an eigenstate of the  $\mathcal{T}$ -invariant Hamiltonian  $H$  or propagator  $U$ , then  $\mathcal{T}\psi$  is an eigenstate of  $H$  or  $U$ , respectively, with the same eigenvalue, too. Therefore each eigenvalue is twofold degenerate—the well known Kramers degeneracy (Messiah 1961).

We turn to the question as to which condition the propagator  $U$  of the spin- $\frac{1}{2}$  kicked rotator (2.5) has to fulfil to be time-reversal invariant with respect to  $\mathcal{T}_\theta$  or  $\mathcal{T}_p$ . First we choose  $\mathcal{T}_\theta$  and show the trivial consequences.

$$\mathcal{T}_\theta U \mathcal{T}_\theta^{-1} = \exp\left(i\frac{p^2}{4\tau}\right) \exp[ik(V_0(\theta) - V(\theta)\sigma)] \exp\left(i\frac{p^2}{4\tau}\right) \quad (4.12)$$

which is equal to  $U^\dagger$  for the trivial case  $V(\theta) = \mathbf{0}$ , only. As was already mentioned at the end of §3, the dynamics then factorises into two identical copies, equivalent to the spin-0 kicked rotator. Kramers’ degeneracy therefore appears trivially.

But if we use  $\mathcal{T}_p$  instead, we find non-trivial conditions for  $\mathcal{T}_p$ -invariance to hold:

$$\mathcal{T}_p U \mathcal{T}_p^{-1} = \exp\left(i\frac{p^2}{4\tau}\right) \exp[ik(V_0(-\theta) - V(-\theta)\sigma)] \exp\left(i\frac{p^2}{4\tau}\right) \tag{4.13}$$

which is equal to  $U^\dagger$  in the case

$$V_0(-\theta) = V_0(\theta) \quad V(-\theta) = -V(\theta). \tag{4.14}$$

To appreciate this symmetry a little better we look at the  $(2 \times 2)$  matrix of the potential  $V = V_0\mathbf{1} + V\sigma$

$$\{V(\theta)^{\rho\sigma}\} = \begin{pmatrix} V_0(\theta) + V_3(\theta) & V_1(\theta) - iV_2(\theta) \\ V_1(\theta) + iV_2(\theta) & V_0(\theta) - V_3(\theta) \end{pmatrix}. \tag{4.15}$$

In this notation (4.14) takes the form

$$V(\theta)^{\rho\sigma} = \rho\sigma V(-\theta)^{-\sigma,-\rho} \quad \rho, \sigma = \pm 1. \tag{4.16a}$$

For reasons of completeness we add the Hermiticity condition

$$V(\theta)^{\rho\sigma} = (V(\theta)^{\sigma\rho})^*. \tag{4.16b}$$

For the Fourier transform of  $V(\theta)^{\rho\sigma}$  this implies

$$\hat{V}_n^{\rho\sigma} = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{in\theta} V^{\rho\sigma}(\theta) = \rho\sigma \hat{V}_{-n}^{-\sigma,-\rho}. \tag{4.17a}$$

The Hermiticity condition (4.16b) takes the form

$$\hat{V}_n^{\rho\sigma} = (\hat{V}_{-n}^{\sigma\rho})^*. \tag{4.17b}$$

Looking at the definition (3.14) of the potential  $W$  one immediately reads off

$$W_0(-\theta) = W_0(\theta) \quad W(-\theta) = -W(\theta). \tag{4.18}$$

As it turns out, the potential  $W$  possesses the same symmetries as  $V$ . Therefore we find for the Fourier transform  $\hat{W}$ , which is connected with the tight binding model via (3.10), the same symmetries as for  $V$  given in (4.17):

$$\hat{W}_n^{\rho\sigma} = \rho\sigma \hat{W}_{-n}^{-\sigma,-\rho} \quad \rho, \sigma = \pm 1. \tag{4.19}$$

This is the well known symplectic symmetry for the tight binding model (Evangelou and Ziman 1988, Zanon and Pichard 1988).

We end this section with two remarks. First we notice that (4.19) implies that for the  $(2 \times 2)$  matrix  $\hat{W}_0$  is of the form  $\hat{W}_0^{1,1}\mathbf{1}$ , and can therefore play the role of an energy in the tight binding equation (3.10), as was already mentioned. Then we look at the mapping between the symplectic rotator and the symplectic tight binding model with diagonal disorder. If the mapping implies their equivalence concerning localisation properties of the eigenfunctions (which is still open to discussion even in the case of the spin-0 kicked rotator) then the investigation of the symplectic rotator might give important clues for the dependence of localisation properties on symmetry.

### 5. The resonant symplectic kicked rotator

We turn again to the resonant case which was shown in §2 to be equivalent to a finite-dimensional problem after fixing the Bloch number  $a$  in the range  $-\pi \leq a < \pi$ . What are the consequences of the time-reversal invariance (with respect to  $\mathcal{F}_p$ , for example) of the propagator  $U$  for the  $(2N \times 2N)$  matrix  $U(a)_{rs}^{\rho\sigma}$  given by equations (2.18) and (2.10)? We look at an eigenvector  $\psi^\sigma$  of this matrix. The Bloch theorem (2.13) written in the  $p$  representation tells us that

$$\psi_{m+N}^\sigma = e^{-ia} \psi_m^\sigma. \tag{5.1}$$

Acting on  $\psi^\sigma$ , the time-reversal operator  $\mathcal{F}_p$  generates a second eigenvector with the same eigenvalue, but orthogonal to the first one. The second eigenvector fulfils the Bloch theorem for a different Bloch number, namely  $-a$ :

$$\mathcal{F}_p^{\rho\sigma} \psi_{m+N}^\sigma = \mathcal{F}_p^{\rho\sigma} e^{-ia} \psi_m^\sigma = e^{ia} \mathcal{F}_p^{\rho\sigma} \psi_m^\sigma. \tag{5.2}$$

Therefore  $\mathcal{F}_p^{\rho\sigma} \psi^\sigma$  is an eigenvector of the same matrix  $U(a)$  only for  $a = 0$ . This implies that only the spectrum of  $U(a = 0)$  shows Kramers' degeneracy. For  $a \neq 0$  each degenerate eigenvalue pair is split into the spectra of  $U(a)$  and  $U(-a)$ . The spectrum of the full, infinite-dimensional eigenvalue problem, embracing the spectra of  $U(a)$  for all real  $a$ , therefore still shows Kramers' degeneracy.

The assumption that  $U$  is  $\mathcal{F}_p$ -invariant implies that  $V = V_0 \mathbf{1} + V \sigma$  fulfils (4.16a), i.e.

$$V(\theta)^{\rho\sigma} = \rho\sigma V(-\theta)^{-\sigma,-\rho} \quad \rho, \sigma = \pm 1 \tag{5.3}$$

which implies the following connection between  $U(a)$  and  $U(-a)$ :

$$\begin{aligned} U(a)_{rs}^{\rho\sigma} &= \exp[-\frac{1}{4}i\pi(r^2 + s^2)] \frac{1}{N} \sum_{j=1}^N \exp[i(r-s)(-\theta_j)] \rho\sigma F(-\theta_j)^{-\sigma,-\rho} \\ &= \rho\sigma U(-a)_{sr}^{-\sigma,-\rho}. \end{aligned} \tag{5.4}$$

As equation (5.4) stands, it poses only a symmetry condition on  $U(a)$  for  $a = 0$ . In the opposite case,  $a \neq 0$ , it connects two different matrices and has therefore no implications for each of their spectra. The only consequence of equation (5.4) is that the spectra of  $U(a)$  and  $U(-a)$  are identical. If, therefore,  $U(a)$  has spectral properties of a random unitary  $(2N \times 2N)$  matrix and does not possess any additional anti-unitary symmetry, then the appropriate ensemble is the unrestricted one, namely CUE, which yields a quadratic degree of level repulsion (Kuś *et al* 1987) between the  $2N$  eigenphases.

Things are different in the case  $a = 0$ . Now equation (5.4) is an anti-unitary symmetry of  $U(0)$ :

$$U(0)_{rs}^{\rho\sigma} = \rho\sigma U(0)_{sr}^{-\sigma,-\rho} \quad \rho, \sigma = \pm 1. \tag{5.5}$$

We again assume that  $U(a = 0)$  has the spectral properties of a random unitary  $(2N \times 2N)$  matrix with anti-unitary symmetries to be specified. The conditions under which this assumption is valid will be checked in the next section. Now we have to take

into account that  $U(0)$  shows Kramers' degeneracy. The degree of repulsion  $\beta$  between different (non-degenerate) eigenphases can take on the values 1, 2 and 4, depending on the codimension of exact degeneracies that almost-degenerate perturbation theory allows to calculate (Kuš *et al* 1987, Scharf *et al* 1988).

To understand this in simple terms we assume that  $U(a = 0)$  commutes with no, one or two  $k$ -independent parity operators, which themselves commute with  $\mathcal{T}$ , anti-commute among themselves and square to  $-1$ . For instance we could think of  $\exp(\frac{1}{2}i\pi\sigma_\nu)$  for  $\nu = 1, 2$  or  $3$ .

First we assume that there is no such parity operator—so the dynamics does not possess any rotational symmetry in spin space. Each Kramers degenerate two-dimensional subspace must be treated, upon increasing the kick strength  $k$ , as one entity. The eigenstates  $\psi$  and  $\mathcal{T}\psi$  are 'intertwined'—the literal meaning of symplectic. Only in this case without any rotational symmetries in spin space we will call the spin- $\frac{1}{2}$  rotator a symplectic rotator. Therefore §4 contains only necessary conditions for the symplectic rotator. For an exact degeneracy between different pairs of Kramers degenerate eigenvalues to become possible, more conditions have to be fulfilled (the codimension is larger) than in the usual case without Kramers' degeneracy. Therefore the eigenvalues show a stronger tendency to repel each other. The degree  $\beta$  of level repulsion turns out to be 4 (Scharf *et al* 1988).

Things change if there exists one parity  $W$  of the kind mentioned. It becomes possible to construct a new time-reversal symmetry  $\mathcal{S} = \mathcal{T}W = W\mathcal{T}$  of  $U$  which squares to  $+1$ . We assume

$$\begin{aligned} \mathcal{T}U\mathcal{T}^{-1} &= U^\dagger & \mathcal{T}^2 &= -1 \\ \mathcal{T}W\mathcal{T}^{-1} &= W & W^2 &= -1 \\ WUW^\dagger &= U \end{aligned} \tag{5.6}$$

and find for the new time-reversal operator  $\mathcal{S}$

$$\mathcal{S}U\mathcal{S}^{-1} = \mathcal{T}WUW^\dagger\mathcal{T}^{-1} = U^\dagger \quad \mathcal{S}^2 = \mathcal{T}W\mathcal{T}^{-1}\mathcal{T}^2W = 1. \tag{5.7}$$

Again almost-degenerate perturbation theory allows one to calculate the codimension of the degeneracies between pairs of different eigenvalues (each of them Kramers degenerate):  $\beta = 2$ . Although the dynamics possesses a time reversal which squares to  $+1$  this does not automatically lead to linear repulsion of the eigenvalues—in contrast to the spin-0 rotator. Upon closer inspection it turns out that  $\mathcal{S}$  is a time-reversal symmetry of  $U(0)$  that allows to split the  $(2N \times 2N)$  matrix  $U(0)$  into two unitary  $(N \times N)$  matrices with identical spectra between which  $\mathcal{S}$  establishes an anti-unitary relation. On the spectra of each of these submatrices the  $\mathcal{S}$ -invariance therefore has no influence—a situation strongly reminiscent of the previous discussion of the spectrum of  $U(a)$  for  $a \neq 0$ . Again, if one of these submatrices has spectral properties of a random unitary  $(N \times N)$  matrix then the appropriate ensemble is the unrestricted one (CUE). Quadratic level repulsion ( $\beta = 2$ ) is the consequence.

Finally, turning to the third case when yet another parity  $\tilde{W}$  exists with the same properties (5.6) as  $W$  and

$$W\tilde{W} + \tilde{W}W = 0 \tag{5.8}$$

then almost-degenerate perturbation theory proves that we have a linear degree of level repulsion:  $\beta = 1$ . A second time-reversal operator  $\tilde{\mathcal{S}} = \mathcal{T}\tilde{W}$  can be constructed, having

the same properties (5.7) as  $\mathcal{S}$  but not commuting with  $\mathcal{S}$ . Therefore  $\tilde{\mathcal{S}}$ -invariance of  $U$  is a second anti-unitary restriction upon  $U(0)$  and the first anti-unitary restriction upon the two mentioned unitary ( $N \times N$ ) submatrices of  $U(0)$ , which are now expected to have COE properties ( $\beta = 1$ ).

Some of these predictions have already been verified for kicked tops (Kuš *et al* 1987, Scharf *et al* 1988). In the next section we check all previous predictions for the spin- $\frac{1}{2}$  rotator numerically.

### 6. Numerical results: the delocalised case

We choose a potential  $V(\theta)^{\rho\sigma}$  which fulfils (4.16) and is of simple form, but nevertheless allows for observing all kinds of symmetries which have been discussed in the previous section:

$$V_0(\theta) = \cos(\theta) \quad V_v(\theta) = v_v \sin(v\theta) \tag{6.1}$$

with real constants  $v_v$  for  $v = 1, 2, 3$ . With this choice, the unitary propagator  $U$ , given by (2.5), is  $\mathcal{F}_p$ -invariant (cf (4.14)).

We choose  $K = 5$  for the classical kick strength with the consequence that the classical motion is strongly chaotic. We then turn to the resonant case (2.8) and choose  $M = 2$  and  $N = 401$ , which guarantees that the eigenfunctions are strongly delocalised over the unperturbed  $2N$ -dimensional basis. For different choices of the coupling constants  $v_v$  we have diagonalised the unitary ( $2N \times 2N$ ) matrix  $U(a)$  given by (2.18).

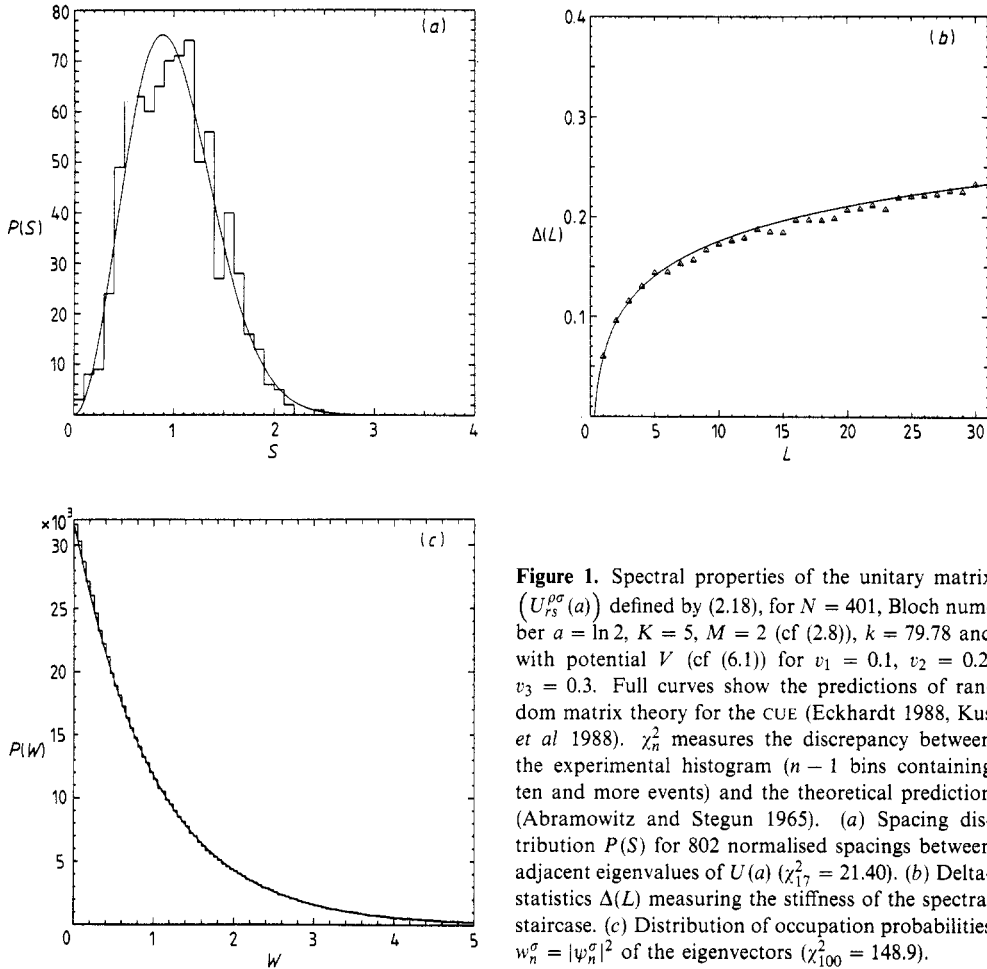
First we choose  $v_v \neq 0$  for all  $v$ . Then the dynamics possesses only  $\mathcal{F}$ -invariances with  $\mathcal{F}^2 = -1$  and we expect to find CUE results for  $a \neq 0$  and CSE results for  $a = 0$ , as was explained in the last section.

Figure 1 shows numerical results for the case  $a \neq 0$ . The spectrum of  $U(a)$  does not possess Kramers' degeneracy. The spacing distribution  $P(S)$ , the so-called  $\Delta$ -statistics  $\Delta(L)$ , which measures the stiffness of the spectrum by calculating the deviation of the 'spectral staircase' from a straight line over an interval of  $L$  mean spacings (see for example Eckhardt 1988), as well as the distribution of squared moduli of the eigenvector components,  $\psi_n^\sigma$ , follow the CUE predictions quite closely.

In figure 2 we see that the same holds true for  $a = 0$  and the CSE predictions, although the deviations of the eigenvector distribution from the theoretical prediction are statistically significant. The reason for this lies in the fact that we analysed the eigenvector components in a very special representation. If we compare two Kramers degenerate eigenstates  $\psi^\rho$  and  $\mathcal{F}_p^{\rho\sigma}\psi^\sigma$  in the  $p$  representation, we find  $|\psi_n^\sigma|^2 = |(\mathcal{F}\psi)_n^{-\sigma}|^2$ . Therefore the occupation probabilities within one Kramers degenerate two-dimensional subspace,  $w_n^\sigma = \frac{1}{2}(|\psi_n^\sigma|^2 + |(\mathcal{F}\psi)_n^\sigma|^2)$ , (which are the appropriate quantities to analyse in the case of Kramers' degeneracy, as they are invariant under arbitrary  $U(2)$  transformations) fulfil the condition:  $w_n^\sigma = w_n^{-\sigma}$ . This symmetry is a restriction upon the  $w_n^\sigma$  and thereby reduces their fluctuations around their mean value  $1/2N$ . This leads to the more pronounced maximum in their histogram in comparison with the theoretical distribution for CSE, which is deduced under the assumption that  $\sum_{n,\sigma} w_n^\sigma = 1$  is the only restriction upon the  $w_n^\sigma$ .

If we set  $v_1 = 0$  then the dynamics possesses a parity  $W_1 = \Pi \exp(\frac{1}{2}i\pi\sigma_1)$  with the following properties

$$\begin{aligned} W_1\theta W_1^\dagger &= \Pi\theta\Pi^{-1} = -\theta & W_1p W_1^\dagger &= \Pi p\Pi^{-1} = -p \\ W_1\sigma_1 W_1^\dagger &= \sigma_1 & W_1\sigma_v W_1^\dagger &= -\sigma_v \quad v = 2, 3. \end{aligned} \tag{6.2}$$



**Figure 1.** Spectral properties of the unitary matrix  $(U_{rs}^{\rho\sigma}(a))$  defined by (2.18), for  $N = 401$ , Bloch number  $a = \ln 2$ ,  $K = 5$ ,  $M = 2$  (cf (2.8)),  $k = 79.78$  and with potential  $V$  (cf (6.1)) for  $v_1 = 0.1$ ,  $v_2 = 0.2$ ,  $v_3 = 0.3$ . Full curves show the predictions of random matrix theory for the CUE (Eckhardt 1988, Kuš *et al* 1988).  $\chi_n^2$  measures the discrepancy between the experimental histogram ( $n - 1$  bins containing ten and more events) and the theoretical prediction (Abramowitz and Stegun 1965). (a) Spacing distribution  $P(S)$  for 802 normalised spacings between adjacent eigenvalues of  $U(a)$  ( $\chi_{17}^2 = 21.40$ ). (b) Delta-statistics  $\Delta(L)$  measuring the stiffness of the spectral staircase. (c) Distribution of occupation probabilities  $w_n^\sigma = |\psi_n^\sigma|^2$  of the eigenvectors ( $\chi_{100}^2 = 148.9$ ).

It clearly follows that

$$W_1 U W_1^\dagger = U \quad W_1^2 = -\mathbf{1}. \tag{6.3}$$

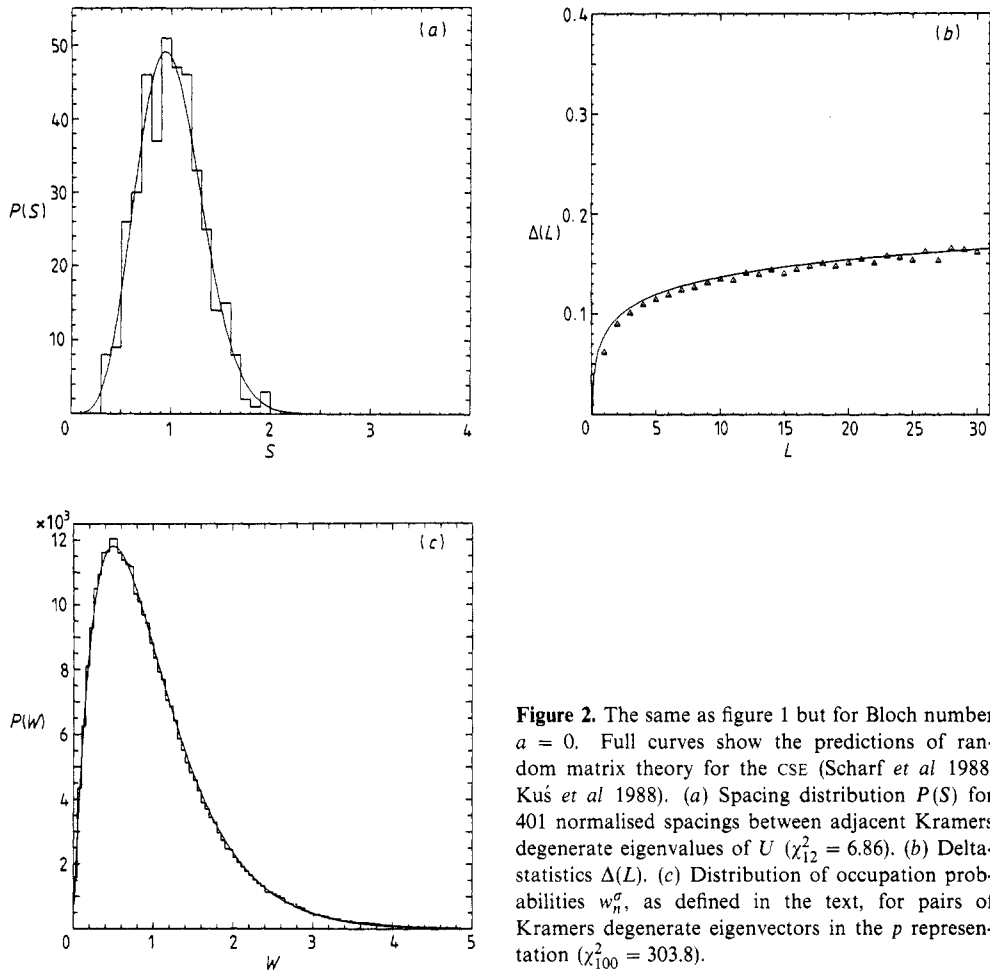
From the discussion in the last section we are led to expect that the spectral statistics of  $U(a = 0)$  are given by CUE. Figure 3 shows that this is actually the case.

If we set  $v_1 = v_2 = 0$  the dynamics possesses other parities, for example  $W_2 = \Pi \exp(\frac{1}{2}i\pi\sigma_2)$ , with the same properties (6.2) and (6.3) as  $W_1$ . In addition (5.8) is fulfilled:

$$W_1 W_2 + W_2 W_1 = \exp(\frac{1}{2}i\pi\sigma_1) \exp(\frac{1}{2}i\pi\sigma_2) + \exp(\frac{1}{2}i\pi\sigma_2) \exp(\frac{1}{2}i\pi\sigma_1) = 0. \tag{6.4}$$

Therefore the spectral statistics of  $U(0)$  should show COE behaviour. Figure 4 confirms this, although the fluctuations of the spacing distribution and the deviations from the theoretical predictions of the  $\Delta$ -statistics are larger in this case than in the previous ones. Kicked tops showed the same feature, too, which is not yet understood.

Finally, if we set  $v_1 = v_2 = v_3 = 0$  the system factorises into two identical copies of the spin-0 kicked rotator. Kramers' degeneracy is fulfilled trivially. Now  $U$



**Figure 2.** The same as figure 1 but for Bloch number  $a = 0$ . Full curves show the predictions of random matrix theory for the CSE (Scharf *et al* 1988, Kuś *et al* 1988). (a) Spacing distribution  $P(S)$  for 401 normalised spacings between adjacent Kramer's degenerate eigenvalues of  $U$  ( $\chi_{12}^2 = 6.86$ ). (b) Delta-statistics  $\Delta(L)$ . (c) Distribution of occupation probabilities  $w_n^\sigma$ , as defined in the text, for pairs of Kramer's degenerate eigenvectors in the  $p$  representation ( $\chi_{100}^2 = 303.8$ ).

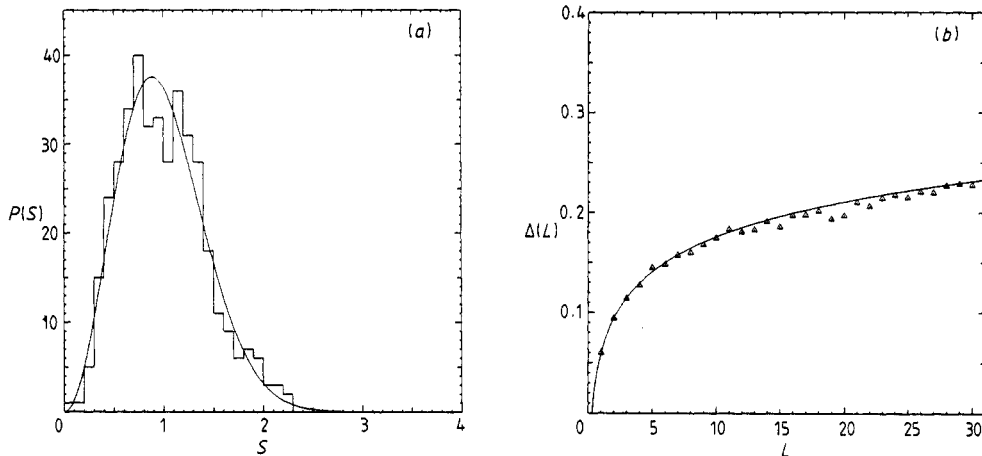
possesses a parity that squares to  $\mathbf{1}$ , namely  $\Pi$ , as was defined in (6.2). To analyse the spectral properties of  $U(a = 0)$  the 'even' and the 'odd' subspaces have to be treated independently.

Besides the features of the spin- $\frac{1}{2}$  rotator that are connected with the Bloch number  $a$  these results are in agreement with other findings for kicked tops (Kuś *et al* 1987, 1988, Scharf *et al* 1988). In the following section we concentrate on those features of the resonant spin- $\frac{1}{2}$  rotator that distinguish it mainly from the kicked tops with half-integer spin, namely localisation features resembling Anderson localisation in tight binding models.

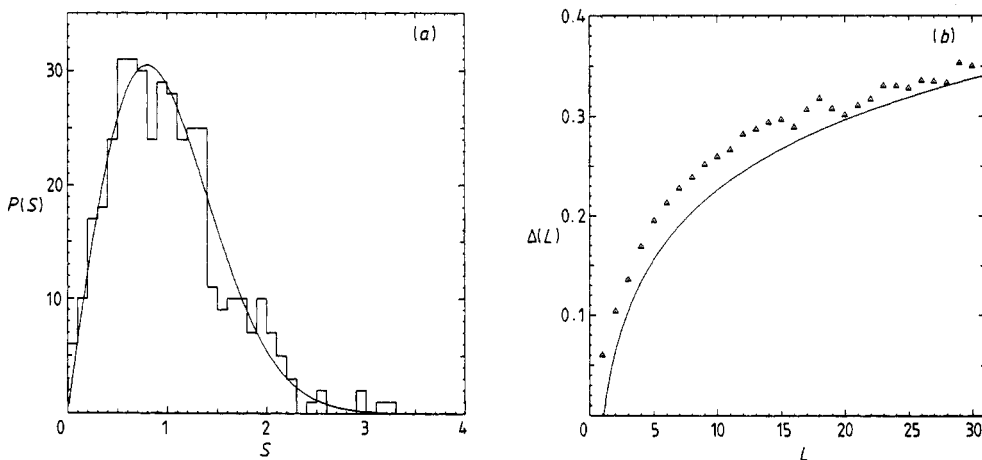
## 7. Numerical results: the localisation–delocalisation transition

One feature of the resonant spin-0 kicked rotator which makes its investigation worthwhile is its ability to show a localisation–delocalisation transition of the eigenfunctions of the unitary propagator in the  $p$  representation. Two recent letters (Izrailev 1988, Casati *et al* 1989b) investigate this feature and find that there exists one main variable





**Figure 3.** The same as figure 1 but for Bloch number  $a = 0$  and coupling  $v_1 = 10^{-6}$ . Full curves show the predictions of random matrix theory for the CUE. (a) Spacing distribution  $P(S)$  for 401 normalised spacings between adjacent Kramers degenerate eigenvalues of  $U$  ( $\chi^2_{14} = 7.24$ ). (b) Delta-statistics  $\Delta(L)$ .

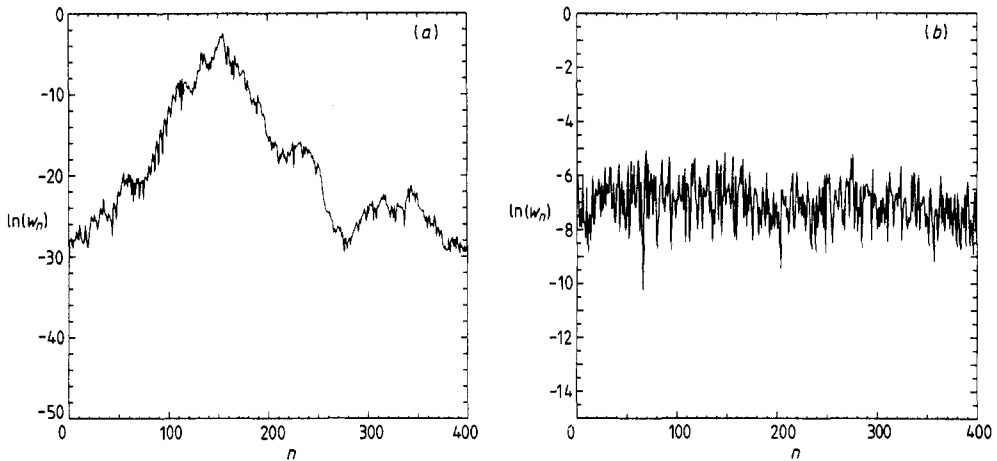


**Figure 4.** The same as figure 1 but for Bloch number  $a = 0$  and couplings  $v_1 = 10^{-6}$ ,  $v_2 = 2v_1$ . Full curves show the predictions of random matrix theory for the COE. (a) Spacing distribution  $P(S)$  for 401 normalised spacings between adjacent Kramers degenerate eigenvalues of  $U$  ( $\chi^2_{18} = 8.99$ ). (b) Delta-statistics  $\Delta(L)$ .

that determines the localisational character of the eigenfunctions, namely  $k^2/N$ . For large values of this variable the eigenfunctions are typically completely delocalised over the  $N$ -dimensional unperturbed basis ( $p$  representation) and random matrix theory explains the results. For decreasing values of  $k^2/N$  the eigenfunctions make a transition to exponential localisation and the spectrum tends to the Poissonian limit ( $\beta = 0$ ) without level repulsion. Moreover, it turned out that after introduction of an appropriate measure for the localisational properties of the (not necessarily exponentially localised) eigenfunctions, scaling behaviour of these properties can be seen with

$k^2/N$  being the only relevant variable (as long as  $k > 1$ ). This scaling behaviour of the spin-0 kicked rotator immediately evokes the question as to whether a similar scaling behaviour can be found for the spin- $\frac{1}{2}$  kicked rotator, too.

To get an impression of how drastically the eigenfunctions change upon changing the kick parameter  $k$  for fixed  $N$ , in figure 5 two typical eigenfunctions of the resonant symplectic rotator (2.18) with the potential (6.1) for  $N = 401$  are shown, one for large, the other for small  $k$ . As it was explained above, we find in the special case of  $\mathcal{F}_p$ -invariance that  $w_n^\sigma = w_n^{-\sigma}$ . Therefore it suffices to show only  $N$  of the  $2N$  probability components. The difference between the two cases is clearly visible: seemingly complete delocalisation of the  $w_n$  for large  $k$  in contrast to exponential localisation for small  $k$ .



**Figure 5.** Occupation probabilities  $w_n^\sigma = w_n^{-\sigma}$ , as defined in the text, for typical eigenfunctions of  $U(a)$  with  $N = 401$ ,  $K = 5$ ,  $v_1 = 0.1$ ,  $v_2 = 0.2$ ,  $v_3 = 0.3$ . (a)  $M = 50$ ,  $k = 3.19$ ,  $k^2/N = 0.025$  (localised case). (b)  $M = 8$ ,  $k = 19.94$ ,  $k^2/N = 0.992$  (delocalised case).

To have a measure for the degree of localisation, we use the definition given by Izrailev (1988) (see also Blümel and Smilansky 1984), which has been applied successfully to show scaling properties of the spin-0 kicked rotator (Casati *et al* 1989b) and which takes the following form in our case:

$$\beta_H = \exp(\mathcal{H} - \mathcal{H}_{2N}) \quad (7.1)$$

with  $\mathcal{H}$  denoting the mean Shannon entropy of the probability distributions  $w_{\alpha,n}$  for all eigenfunction pairs  $\psi_\alpha$ ,  $\mathcal{F}\psi_\alpha$ , of the  $(2N \times 2N)$  matrix  $U(a)$ :

$$\mathcal{H} = -\frac{1}{N} \sum_{\alpha=1}^N \sum_{n=1}^N \sum_{\sigma=\pm 1} w_{\alpha,n}^\sigma \ln(w_{\alpha,n}^\sigma). \quad (7.2)$$

$\mathcal{H}_{2N}$  denotes the entropy  $\mathcal{H}$  for the special case of random eigenvectors with a distribution given by random matrix theory for the CSE of unitary  $(2N \times 2N)$  matrices (Kus *et al* 1988)

$$\mathcal{H}_{2N} = \ln(4N) + \gamma - \frac{3}{2} + (1/4N) + O(1/N^2) \quad (7.3)$$

with  $\gamma = 0.5772\dots$  denoting Euler's constant. With this definition we typically find  $0 \leq \beta_H \leq 1$  with the two extremes of strongly localised and completely delocalised eigenfunctions.

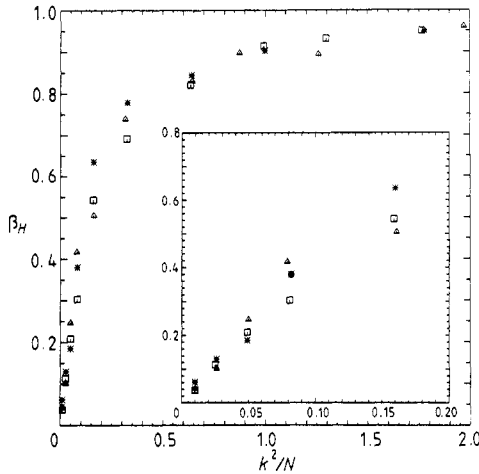
In the case of exponentially localised eigenfunctions with  $w_n^\sigma$  of the form

$$w_n^\sigma = Z^{-1} \exp(-2|n - n_0|/l) \quad Z = 2 \sum_{n=-\infty}^{\infty} \exp(-2|n - n_0|/l) \quad (7.4)$$

we find for  $1 \ll l \ll N$

$$\mathcal{H}(l) = 1 + \ln(2l) + O(1/l) \quad \beta_H(l) \simeq \frac{l}{4N} \exp(\frac{5}{2} - \gamma). \quad (7.5)$$

With the help of this relation it can then be checked numerically whether, for example, the localisation length  $l$  depends linearly on  $k^2/N$  as is the case for the spin-0 rotator for  $l/N$  small enough.



**Figure 6.** Delocalisation  $\beta_H$  (7.1) plotted against localisation parameter  $k^2/N$  for eigenfunctions of  $U(a)$  as in figure 5 but for different values of  $N$  and  $k$ :  $N = 101$  (\*),  $N = 199$  ( $\Delta$ ), and  $N = 401$  ( $\square$ ). The box shows a magnification of the region of small  $k^2/N$  values.

Figure 6 shows the dependence of  $\beta_H$  upon  $k^2/N$  for  $N = 101, 199$  and  $401$  and for different values of  $k$ . Although we expect an exact scaling law to hold only in the limit  $N \rightarrow \infty$ , even for these small values of  $N$  the scaling features are already visible. Comparing these results for the spin- $\frac{1}{2}$  rotator with previous ones for the spin-0 standard rotator (Casati *et al* 1989b) two differences are obvious. In the spin- $\frac{1}{2}$  case the limit of strong delocalisation (CSE limit) is reached much faster, upon increasing  $k^2/N$ , than the strong disorder limit for the spin-0 rotator (COE/CUE limit). The slope of the presumptive theoretical scaling curve at  $k^2/N = 0$  is much larger for the spin- $\frac{1}{2}$  rotator than for the spin-0 rotator. From figure 6 we infer that a linear relation holds between  $\beta_H$  and  $k^2/N$  for small values of  $k^2/N$ , which leads to the following result for the localisation length

$$l = \alpha k^2 \quad (7.6)$$

with the numerical constant  $\alpha \simeq 1.2$ , which is larger by a factor of about 4 than the corresponding value for the spin-0 kicked rotator (Shepelyansky 1986).

## 8. Discussion

We have shown that the spin- $\frac{1}{2}$  kicked rotator is the ideal tool to investigate the influence of anti-unitary symmetries on localisation properties. Its anti-unitary symmetries are easy to control and to interpret physically, and we have given a classification for them which predicts the correct statistical properties for eigenvalues and eigenvectors of the resonant strongly kicked spin- $\frac{1}{2}$  rotator. It was shown that the spin- $\frac{1}{2}$  rotator can be mapped to a tight binding model for a spin- $\frac{1}{2}$  particle moving in a potential with diagonal disorder (for irrational  $\tau/\pi$ ) and with spin coupling terms. Finally it was shown that the localisation properties of the eigenfunctions of the symplectic rotator for rational  $\tau/\pi$  show a scaling behaviour similar to the eigenfunctions of the resonant spin-0 rotator. The most prominent difference between them was that the eigenfunctions of the symplectic kicked rotator are much more delocalised than the eigenfunctions of the spin-0 standard rotator for comparable kick strength. Whether the anti-unitary symmetries or the functional form of the kicking potential are responsible for this difference has to be investigated for the spin-0 as well as for the spin- $\frac{1}{2}$  in greater detail.

The results concerning the scaling behaviour of the kicked rotator for a spin- $\frac{1}{2}$  particle and for a spin-0 particle (Casati *et al* 1989b) should be compared with previous numerical results by Feingold *et al* (1985, 1987, 1988), by Frahm and Mikeska (1988a, b) and by Izrailev (1986) for the spectrum of the resonant spin-0 standard rotator. They clearly showed that there exists a transition from Poissonian behaviour connected with exponentially localised eigenfunctions to Wigner-type behaviour as soon as the localisation length  $l_\infty$  for eigenfunctions in the infinite system ( $N \rightarrow \infty$ ) becomes comparable to the size  $N$  of the finite system. As  $l_\infty \propto k^2$  (Shepelyansky 1986) the scaling variable  $k^2/N$  is proportional to the ratio between localisation length and the size of the system. Therefore the present scaling results for the spin- $\frac{1}{2}$  and similar scaling results for the spin-0 rotator (Casati *et al* 1989b) concerning the eigenfunctions and previous results concerning eigenvalues fit well into a coherent picture.

It might also be interesting to study the influence of the symplectic symmetry in the non-resonant case ( $\tau/\pi$  irrational). This can be investigated conveniently with the help of the time dependent problem, for which we have given the quantum map. With a trick used recently by Casati *et al* (1989a) to investigate the Anderson transition in a (pseudo) three-dimensional tight binding model it seems to be possible to investigate localisation properties of the two-dimensional symplectic kicked rotator numerically without much effort and to compare it with the two-dimensional spin-0 rotator (Doron and Fishman 1988). Of most interest is the question of whether the two-dimensional symplectic rotator shows an Anderson transition, as is claimed to exist for a symplectic two-dimensional tight binding model (Evangelou and Ziman 1988).

The localisation–delocalisation transition that we have investigated in this paper was for Bloch states (which are delocalised over the whole  $p$  axis) within one lattice period of length  $N$ . This transition is accompanied by a change in the statistical properties of the eigenvalues of the dynamics. The degree  $\beta$  of level repulsion varies between a maximal value  $\beta_0$ , given by random matrix theory, and 0, the Poissonian limit. In the case of the symplectic rotator  $\beta_0 = 4$  and therefore  $\beta$  can take on the values 1 and 2, for example. Numerical results indicate that these intermediate situations can be described by random matrix theory only concerning the statistical properties of the spectra, but not of the eigenvector components. With the help of a recently proposed (Izrailev 1988) and thoroughly tested (Izrailev and Scharf 1989) fitting formula for

intermediate spacing distributions, an experimental value for  $\beta$  can be determined from spacing histograms and compared with  $\beta_H\beta_0$ . Within error bounds, the two quantities agree, which is in favour of the claim that the statistics of the eigenvalues show a similar scaling behaviour as  $\beta_H$ , which reflects properties of the eigenvectors.

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